Problem 1 – Townsend 10.12  Expectation values are constant in time in an energy eigenstate. Hence
\[ \frac{d\langle \mathbf{r} \cdot \mathbf{p} \rangle}{dt} = \frac{i}{\hbar} \langle E | [\hat{H}, \mathbf{r} \cdot \mathbf{p}] | E \rangle = 0 \]
Use this result to show for the Hamiltonian
\[ \hat{H} = \frac{\hat{p}^2}{2\mu} + V(|\mathbf{r}|) \]
that
\[ \langle K \rangle = \langle \frac{\hat{p}^2}{2\mu} \rangle = \frac{1}{2} \langle \mathbf{r} \cdot \nabla V(r) \rangle \]
which can be considered a quantum statement of the virial theorem.

Problem 2 – Townsend 10.14  Suppose that nucleons within the nucleus are presumed to move independently in a potential energy well in the form of an isotropic harmonic oscillator. What are the first five nuclear “magic numbers” within such a model?

Problem 3 – Townsend 10.16  Consider the Hamiltonian for the two-dimensional motion of a particle of mass \( \mu \) in a harmonic oscillator potential:
\[ \hat{H} = \frac{\hat{p}_x^2}{2\mu} + \frac{1}{2} \mu \omega^2 x^2 + \frac{\hat{p}_y^2}{2\mu} + \frac{1}{2} \mu \omega^2 y^2 \]
(a) Show that the energy eigenvalues are given by \( E_n = (n + 1)\hbar \omega \), where \( n = n_1 + n_2 \), with \( n_1, n_2 = 0, 1, 2, \ldots \)
(b) Express the operator \( \hat{L}_z = \hat{x} \hat{p}_y - \hat{y} \hat{p}_x \) in terms of the lowering operators
\[ \hat{a}_1 = \sqrt{\frac{\mu \omega}{2\hbar}} \left( \hat{x} + \frac{i}{\mu \omega} \hat{p}_x \right) \]
and \( \hat{a}_2 = \sqrt{\frac{\mu \omega}{2\hbar}} \left( \hat{y} + \frac{i}{\mu \omega} \hat{p}_y \right) \)
and the corresponding raising operators \( \hat{a}^\dagger_1 \) and \( \hat{a}^\dagger_2 \). Give a symmetry argument showing that \( [\hat{H}, \hat{L}_z] = 0 \). Evaluate this commutator directly and confirm that it indeed vanishes.
(c) Determine the correct linear combination of the energy eigenstates with energy \( E_1 = 2\hbar \omega \) that are eigenstates of \( \hat{L}_z \) by diagonalizing the matrix representation of \( \hat{L}_z \) restricted to this subspace of states.

Problem 4 – Townsend 10.17  The spherically symmetric potential energy of a particle of mass \( \mu \) is given by
\[ V(r) = \begin{cases} 0 & a < r < b \\ \infty & \text{elsewhere} \end{cases} \]
where \( r = \sqrt{x^2 + y^2 + z^2} \).
(a) Determine the ground-state energy.

(b) What is the ground-state position-space eigenfunction up to an overall normalization constant? What condition would you impose to determine this constant?

(c) What is the energy of the first-excited $l = 0$ state? Explain why it would not be so straightforward to determine the energy of the $l = 1$ states.